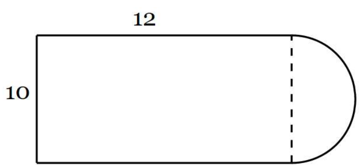
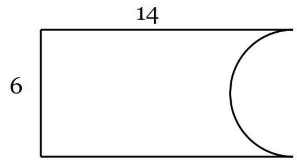
**K. Area and Volume – Assignment**

1. Find the areas of the figures below. All are composed of rectangles and semicircles.

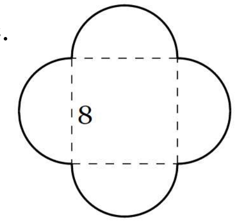
a.




b.

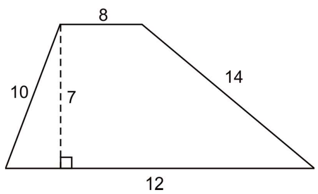



c.

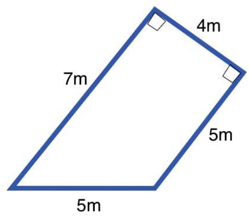



2. Find the areas of the trapezoids pictured below.

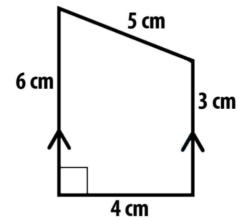
a.




b.

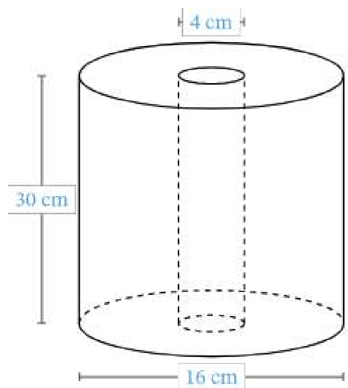



c.

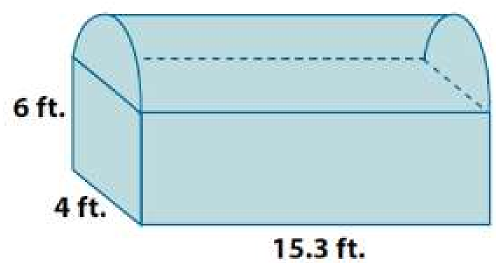



3. Find the volumes of the following figures.

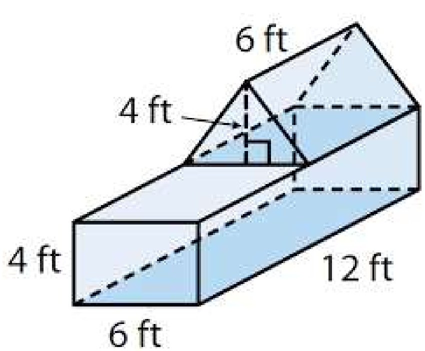
a. A cylindrical paper towel roll with a hollow center



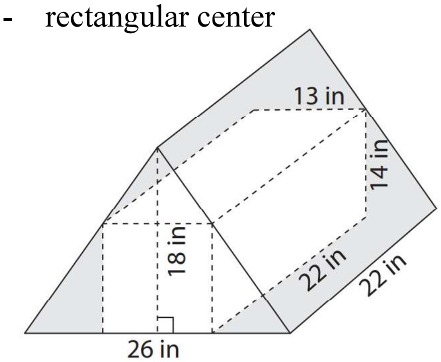

b. A rectangular prism with a semi-cylinder on top of it.




c. A rectangular prism with a triangular prism on top of it




d. A triangular prism with a hollow rectangular center



**L. Adding Fractions and Solving Fractional Equations**

There are two major problem types with fractions: Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one* … in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD*. When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1. a. Combine: $\frac{x}{3}-\frac{x}{4}$

LCD: 12 $\quad \frac{x}{3}\left(\frac{4}{4}\right)-\frac{x}{4}\left(\frac{3}{3}\right)$

$\frac{4x-3x}{12}=\frac{x}{12}$

b. Solve: $\frac{x}{3}-\frac{x}{4}=12$

$12\left(\frac{x}{3}\right)-12\left(\frac{x}{4}\right)=12(12)$

$4x-3x=144 \Rightarrow x=144$

$x=144: \frac{144}{3}-\frac{144}{4}=48-36=12$

2. a. Combine $x+\frac{6}{x}$

LCD: $x \quad x\left(\frac{x}{x}\right)+\frac{6}{x}$

$\frac{x^2+6}{x}$

b. Solve: $x+\frac{6}{x}=5$

$x(x)+x\left(\frac{6}{x}\right)=5x$

$x^2+6=5x \Rightarrow x^2-5x+6=0$

$(x-2)(x-3)=0 \Rightarrow x=2,\ x=3$

$x=2: 2+\frac{6}{2}=2+3=5 \quad x=3: 3+\frac{6}{3}=3+2=5$

3 . a. Combine: $\frac{12}{x+2}-\frac{4}{x}$

LCD: $x(x+2) \quad \left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right)-\frac{4}{x}\left(\frac{x+2}{x+2}\right)$

$\frac{12x-4x-8}{x(x+2)}$

$\frac{8x-8}{x(x+2)}$

b. Solve $\frac{12}{x+2}-\frac{4}{x}=1$

$\frac{12}{x+2}(x)(x+2)-\frac{4}{x}(x)(x+2)=1(x)(x+2)$

$12x-4x-8=x^2+2x \Rightarrow x^2-6x+8=0$

$(x-2)(x-4)=0 \Rightarrow x=2,4$

$x=2: \frac{12}{4}-\frac{4}{2}=3-2=1 \quad x=4: \frac{12}{6}-\frac{4}{4}=2-1=1$

4. a. $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}$

LCD: $2(x-3)^2$

$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right)-\frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$

$\frac{x^2-3x-6}{2(x-3)^2}$

b. Solve $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}=\frac{x-2}{3x-9}$

$\left[\frac{x}{2(x-3)}-\frac{3}{(x-3)^2}=\frac{x-2}{3(x-3)}\right]6(x-3)^2$

$3x(x-3)-18=2(x-3)(x-2)$

$3x^2-9x-18=2x^2-10x+12$

$x^2+x-30=0 \Rightarrow (x+6)(x-5)=0 \Rightarrow x=-6,5$

$x=-6: \frac{-6}{-18}-\frac{3}{81}=\frac{-8}{-27} \quad x=5: \frac{5}{4}-\frac{3}{4}=\frac{3}{6}$

**L. Adding Fractions and Solving Fractional Equations - Assignment**

1. a. Combine: $\frac{2}{3}-\frac{1}{x}$

   b. Solve: $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$

2. a. Combine: $\frac{1}{x-3}+\frac{1}{x+3}$

   b. Solve: $\frac{1}{x-3}+\frac{1}{x+3}=\frac{10}{x^2-9}$

3. a. Combine: $\frac{5}{2x}-\frac{5}{3x+15}$

   b. Solve: $\frac{5}{2x}-\frac{5}{3(x+5)}=\frac{5}{x}$

4. a. Combine: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}$

   b. Solve: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}=\frac{x^2+11}{2x^2-x-1}$

**M. Solving Absolute Value Equations**

Absolute value equations crop up in calculus, especially in BC calculus. The definition of the absolute value function is a piecewise function: $f(x)=|x|=\begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. So, to solve an absolute value equation, split the absolute value equation into two equations, one with a positive parentheses and the other with a negative parentheses and solve each equation. It is possible that this procedure can lead to incorrect solutions so solutions must be checked.

• Solve the following equations.

1. $|x-1|=3$

$x-1=3$, $x=4$

$-(x-1)=3$, $-x+1=3$, $x=-2$

2. $|3x+2|=9$

$3x+2=9$, $3x=7$, $x=\frac{7}{3}$

$-(3x+2)=9$, $-3x-2=9$, $3x=-11$, $x=\frac{-11}{3}$

3. $|2x-1|-x=5$

$2x-1-x=5$, $x=6$

$-(2x-1)-x=5$, $-3x=4$, $x=\frac{-4}{3}$

4. $|x+5|+5=0$

$x+5+5=0$, $x=-10$

$-(x+5)+5=0$, $-x-5+5=0$, $x=0$

Both answers are invalid. It is impossible to add 5 to an absolute value and get 0.

5. $|x^2-x|=2$

$(x^2-x)=2$, $x^2-x-2=0$, $(x-2)(x+1)=0$, $x=2, x=-1$

$-(x^2-x)=2$, $-x^2+x=2$, $0=x^2+x+2$, No real solution

Both solutions check

6. $|x-10|=x^2-10x$

$x-10=x^2-10x$, $x^2-11x+10=0$, $(x-1)(x-10)=0$, $x=1, x=10$

$-(x-10)=x^2-10x$, $-x+10=x^2-10x$, $x^2-9x-10=0$, $(x-10)(x+1)=0$, $x=10, x=-1$

Of the three solutions, only $x=-1$ and $x=10$ are valid.

7. $|x|+|2x-2|=8$

$x+2x-2=8$, $3x=10$, $x=\frac{10}{3}$

$-x+2x-2=8$, $x=10$

$x-(2x-2)=8$, $-x=6$, $x=-6$

$-x-(2x-2)=8$, $-3x=6$, $x=-2$

Of the four solutions, only $x=\frac{10}{3}$ and $x=-2$ are valid

## M. Solving Absolute Value Equations - Assignment

• Solve the following equations.

1. $4|x+8|=20$

2. $|1-7x|=13$

3. $|8+2x|+2x=40$

4. $|4x-5|+5x+2=0$

5. $|x^2-2x-1|=7$

6. $|12-x|=x^2-12x$

7. $|x|+|4x-4|+x=14$

## N. Solving Inequalities

You may think that solving inequalities are just a matter of replacing the equal sign with an inequality sign. In reality, they can be more difficult and are fraught with dangers. And in calculus, inequalities show up more frequently than solving equations. Solving inequalities are a simple matter if they are based on linear equations. They are solved exactly like linear equations, remembering that if you multiply or divide both sides by a negative number, the direction of the inequality sign must be reversed.

However, if the inequality is more complex than a linear function, it is advised to bring all terms to one side. Pretend for a moment it is an equation and solve. Then create a number line which determines whether the transformed inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the zeroes are included or not.

If the inequality involves an absolute value, create two equations, replacing the absolute value with a positive parentheses and a negative parentheses and the inequality sign with an equal sign. Solve each, placing each solution on your number line. Then determine which intervals satisfy the original inequality.

If the inequality involves a rational function, set both numerator and denominator equal to zero, which will give you the values you need for your number line. Determine whether the inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the endpoints are included or not.

• Solve the following inequalities.

1. $2x-8 \le 6x+2$

$-10 \le 4x$ or $-4x \le 10$

$\dfrac{-5}{2} \le x$ $\quad x \ge \dfrac{-5}{2}$

2. $1-\dfrac{3x}{2} > x-5$

$2-3x > 2x-10$

$12 > 5x \Rightarrow x < \dfrac{12}{5}$

3. $-7 \le 6x-1 < 11$

$-6 \le 6x \le 12$

$-1 \le x \le 2$

4. $|2x-1| \le x+4$

$|2x-1|-x-4 \le 0$

$2x-1-x-4=0$ $\quad -2x+1-x-4=0$

$x=5$ $\quad x=-1$

$++++++\!+0---------0+++++++$

-1 5

So $-1 \le x \le 5$ or $[-1,5]$

5. $x^2-3x > 18$

$x^2-3x-18>0 \Rightarrow (x+3)(x-6)>0$

For $(x+3)(x-6)=0$, $x=-3, x=6$

$++++++0-------0+++++++$

-3 6

So $x<-3$ or $x>6$ or $(-\infty,-3)\cup(6,\infty)$

6. $\dfrac{2x-7}{x-5} \le 1$

$\dfrac{2x-7}{x-5}-1=0 \Rightarrow \dfrac{2x-7}{x-5}-\dfrac{x-5}{x-5}<0 \Rightarrow \dfrac{x-2}{x-5}<0$

$++++++0-------\infty+++++++$

2 5

So $2 \le x < 5$ or $[2,5)$

7. Find the domain of $\sqrt{32-2x^2}$

$2(4+x)(4-x) \ge 0$

$------0+++++++0-------$

-4 4

So $-4 \le x \le -4$ or $[-4,4]$

## N. Solving Inequalities - Assignment

• Solve the following inequalities.

1. $5(x-3) \leq 8(x+5)$

2. $4 - \frac{5x}{3} > -\left(2x + \frac{1}{2}\right)$

3. $\frac{3}{4} > x + 1 > \frac{1}{2}$

4. $x + 7 \geq |5 - 3x|$

5. $(x+2)^2 < 25$

6. $x^3 < 4x^2$

7. $\frac{5}{x-6} \geq \frac{1}{x+2}$

8. Find the domain of $\sqrt{\frac{x^2 - x - 6}{x - 4}}$

**O. Exponential Functions and Logarithms**

Calculus spends a great deal of time on exponential functions in the form of $b^x$. Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If $y = b^x$ then $x = \log_b y$. So when you are trying to find the value of $\log_2 32$, state that $\log_2 32 = x$ and $2^x = 32$ and therefore $x = 5$.

If the base of a log statement is not specified, it is defined to be 10. When we asked for log 100, we are solving the equation: $10^x = 100$ and $x = 2$. The function $y = \log x$ has domain $(0,\infty)$ and range $(-\infty,\infty)$. In calculus, we primarily use logs with base *e*, which are called natural logs (ln). So finding ln 5 is the same as solving the equation $e^x = 5$. Students should know that the value of $e = 2.71828\ldots$

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

i. $\log a + \log b = \log(a \cdot b)$  ii. $\log a - \log b = \log\left(\dfrac{a}{b}\right)$  iii. $\log a^b = b \log a$

1. Find a. $\log_4 8$

$$\log_4 8 = x$$
$$4^x = 8 \Rightarrow 2^{2x} = 2^3$$
$$x = \frac{3}{2}$$

b. $\ln\sqrt{e}$

$$\ln\sqrt{e} = x$$
$$e^x = e^{1/2}$$
$$x = \frac{1}{2}$$

c. $10^{\log 4}$

$$\log 4 = x$$
$$10^x = 4 \text{ so } 10^{\log 4} = 4$$
10 to a power and log are inverses

d. $\log 2 + \log 50$

$$\log(2 \cdot 50) = \log 100$$
2

e. $\log_4 192 - \log_4 3$

$$\log_4\left(\frac{192}{3}\right)$$
$$\log_4 64 = 3$$

f. $\ln\sqrt[5]{e^3}$

$$\ln e^{3/5} = \frac{3}{5}\ln e = \frac{3}{5}$$

2. Solve a. $\log_9\left(x^2 - x + 3\right) = \dfrac{1}{2}$

$$x^2 - x + 3 = 9^{1/2}$$
$$x(x-1) = 0$$
$$x = 0, x = 1$$

b. $\log_{36} x + \log_{36}(x-1) = \dfrac{1}{2}$

$$\log_{36} x(x-1) = \frac{1}{2}$$
$$x(x-1) = 36^{1/2} = 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
Only $x = 3$ is in the domain

c. $\ln x - \ln(x-1) = 1$

$$\ln\left(\frac{x}{x-1}\right) = 1$$
$$\frac{x}{x-1} = e \Rightarrow x = ex - e$$
$$x = \frac{e}{e-1}$$

d. $5^x = 20$

$$log_5(5^x) = log_5(20)$$
$$x = log_5(20)$$
$$x = 1.861$$

e. $e^{-2x} = 5$

$$\ln e^{-2x} = \ln 5$$
$$-2x = \ln 5 \Rightarrow x = \frac{-\ln 5}{2}$$

**O. Exponential Functions and Logarithms - Assignment**

1. Find a. $\log_2 \frac{1}{4}$ 

b. $\log_8 4$ 

c. $\ln \frac{1}{\sqrt[3]{e^2}}$

d. $5^{\log_5 40}$ 

e. $e^{\ln 12}$ 

f. $\log_{12} 2 + \log_{12} 9 + \log_{12} 8$

g. $\log_2 \frac{2}{3} + \log_2 \frac{3}{32}$ 

h. $\log_{\frac{1}{3}} \frac{4}{3} - \log_{\frac{1}{3}} 12$ 

i. $\log_3 \left(\sqrt{3}\right)^5$

2. Solve a. $\log_5 (3x-8) = 2$ 

b. $\log_9 \left(x^2 - x + 3\right) = \frac{1}{2}$ 

c. $\log(x-3) + \log 5 = 2$

d. $\log_2 (x-1) + \log_2 (x+3) = 5$ 

e. $\log_5 (x+3) - \log_5 x = 2$ 

f. $\ln x^3 - \ln x^2 = \frac{1}{2}$

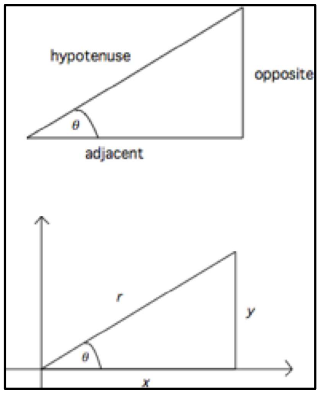
g. $3^{x-2} = 18$ 

h. $e^{3x+1} = 10$

## P. Right Angle Trigonometry

Trigonometry is an integral part of AP calculus. Students must know the basic trig function definitions in terms of opposite, adjacent and hypotenuse as well as the definitions if the angle is in standard position.

Given a right triangle with one of the angles named $\theta$, and the sides of the triangle relative to $\theta$ named opposite ($y$), adjacent ($x$) , and hypotenuse ($r$) we define the 6 trig functions to be:




$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyptotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyptotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

The Pythagorean theorem ties these variables together: $x^2 + y^2 = r^2$. Students should recognize right triangles with integer sides: 3-4-5, 5-12-13, 8-15-17, 7-24-25. Also any multiples of these sides are also sides of a right triangle. Since $r$ is the largest side of a right triangle, it can be shown that the range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$, the range of $\csc\theta$ and $\sec\theta$ is $(-\infty,-1]\cup[1,\infty)$ and the range of $\tan\theta$ and $\cot\theta$ is $(-\infty,\infty)$.

Also vital to master is the signs of the trig functions in the four quadrants. A good way to remember this is A – S – T – C where All trig functions are positive in the 1st quadrant, Sin is positive in the 2nd quadrant, Tan is positive in the 3rd quadrant and Cos is positive in the 4th quadrant.

1. Let be a point on the terminal side of $\theta$. Find the 6 trig functions of $\theta$. (Answers need not be rationalized).

a) $P(-8,6)$

$x=-8, y=6, r=10$

$\sin\theta = \frac{3}{5}$ $\quad \csc\theta = \frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\quad \sec\theta = -\frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ $\quad \cot\theta = -\frac{4}{3}$

b. $P(1,3)$

$x=1,\ y=3,\ r=\sqrt{10}$

$\sin\theta = \frac{3}{\sqrt{10}}$ $\quad \csc\theta = \frac{\sqrt{10}}{3}$

$\cos\theta = \frac{1}{\sqrt{10}}$ $\quad \sec\theta = \sqrt{10}$

$\tan\theta = 3$ $\quad \cot\theta = \frac{1}{3}$

c. $P(-\sqrt{10},-\sqrt{6})$

$x=-\sqrt{10},\ y=-\sqrt{6},\ r=4$

$\sin\theta = -\frac{\sqrt{6}}{4}$ $\quad \csc\theta = -\frac{4}{\sqrt{6}}$

$\cos\theta = -\frac{\sqrt{10}}{4}$ $\quad \sec\theta = -\frac{4}{\sqrt{10}}$

$\tan\theta = \sqrt{\frac{3}{5}}$ $\quad \cot\theta = \sqrt{\frac{5}{3}}$

2. If $\cos\theta = \frac{2}{3}$, $\theta$ in quadrant IV, find $\sin\theta$ and $\tan\theta$

$x=2,\ r=3,\ y=-\sqrt{5}$

$\sin\theta = -\frac{\sqrt{5}}{3}, \tan\theta = -\frac{\sqrt{5}}{2}$

3. If $\sec\theta = \sqrt{3}$ find $\sin\theta$ and $\tan\theta$

$\theta$ is in quadrant I or IV

$x=1,\ y=\pm\sqrt{2},\ r=\sqrt{3}$

$\sin\theta = \pm\sqrt{\frac{2}{3}},\ \tan\theta = \pm\sqrt{2}$

4. Is $3\cos\theta + 4 = 2$ possible?

$3\cos\theta = -2$

$\cos\theta = -\frac{2}{3}$ which is possible.

**P. Right Angle Trigonometry - Assignment**

1. Let be a point on the terminal side of $\theta$. Find the 6 trig functions of $\theta$. (Answers need not be rationalized).

a) $P(15,8)$ b. $P(-2,3)$ c. $P(-2\sqrt{5},-\sqrt{5})$

2. If $\tan\theta = \frac{12}{5}$, $\theta$ in quadrant III, find $\sin\theta$ and $\cos\theta$

3. If $\csc\theta = \frac{6}{5}$, $\theta$ in quadrant II, find $\cos\theta$ and $\tan\theta$

4. $\cot\theta = \frac{-2\sqrt{10}}{3}$ find $\sin\theta$ and $\cos\theta$

5. Find the quadrants where the following is true: Explain your reasoning.

a. $\sin\theta > 0$ and $\cos\theta < 0$ b. $\csc\theta < 0$ and $\cot\theta > 0$ c. all functions are negative

6. Which of the following is possible? Explain your reasoning.

a. $5\sin\theta = -2$ b. $3\sin\alpha + 4\cos\beta = 8$ c. $8\tan\theta + 22 = 85$

**Q. Special Angles**

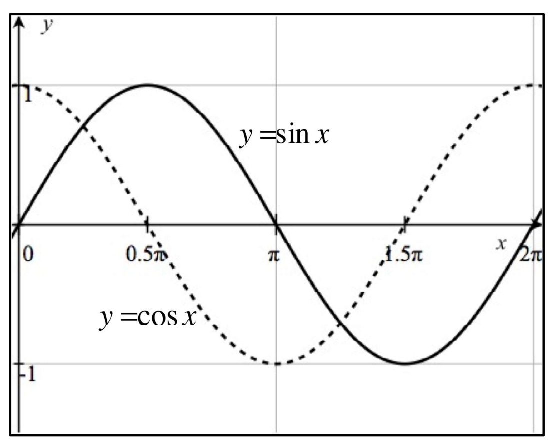
Students must be able to find trig functions of quadrant angles $(0,\ 90°, 180°, 270°)$ and special angles, those based on the $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles.

First, for most calculus problems, angles are given and found in radians. Students must know how to convert degrees to radians and vice-versa. The relationship is $2\pi$ radians = 360° or $\pi$ radians = 180°. Angles are assumed to be in radians so when an angle of $\dfrac{\pi}{3}$ is given, it is in radians. However, a student should be able to picture this angle as $\dfrac{180°}{3} = 60°$. It may be easier to think of angles in degrees than radians, but realize that unless specified, angle measurement must be written in radians. For instance, $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$.

The trig functions of **quadrant angles** $\left(0,\ 90°, 180°, 270° \text{ or } 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}\right)$ can quickly be found. Choose a point along the angle and realize that *r* is the distance from the origin to that point and always positive. Then use the definitions of the trig functions.

| $\theta$ | point | $x$ | $y$ | $r$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $(1,0)$ | 1 | 0 | 1 | 0 | 1 | 0 | does not exist | 1 | does not exist |
| $\frac{\pi}{2}$ or 90° | $(0,1)$ | 0 | 1 | 1 | 1 | 0 | does not exist | 1 | does not exist | 0 |
| $\pi$ or 180° | $(-1,0)$ | -1 | 0 | 1 | 0 | -1 | 0 | does not exist | -1 | does not exist |
| $\frac{3\pi}{2}$ or 270° | $(0,-1)$ | 0 | -1 | 1 | -1 | 0 | Does not exist | -1 | does not exist | 0 |

If you picture the graphs of $y = \sin x$ and $y = \cos x$ as shown to the right, you need not memorize the table. You must know these graphs backwards and forwards.



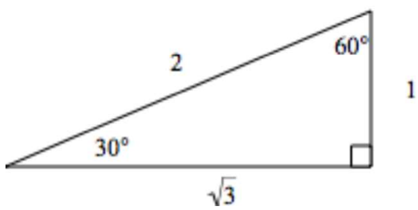

• Without looking at the table, find the value of

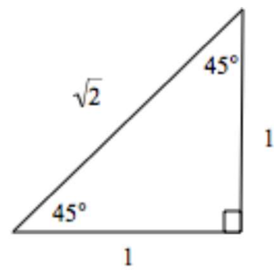
a. $5\cos 180° - 4\sin 270°$

$$\boxed{\begin{array}{l} 5(-1) - 4(-1) \\ -5 + 4 = -1 \end{array}}$$

b. $\left(\dfrac{8\sin\frac{\pi}{2} - 6\tan\pi}{5\sec\pi - \csc\frac{3\pi}{2}}\right)^2$ $\boxed{\left[\dfrac{8(1) - 6(0)}{5(-1) - (-1)}\right]^2 = \left(\dfrac{8}{-4}\right)^2 = 4}$

Because over half of the AP exam does not use a calculator, you must be able to determine trig functions of **special angles**. You must know the relationship of sides in both $30°-60°-90°$ $\left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ and $45°-45°-90°$ $\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangles.




In a $30°-60°-90°$ $\left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ triangle, the ratio of sides is $1-\sqrt{3}-2$.

In a $45°-45°-90°$ $\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangle, the ratio of sides is $1-1-\sqrt{2}$.

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $30°\left(\text{or } \frac{\pi}{6}\right)$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $45°\left(\text{or } \frac{\pi}{4}\right)$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| $60°\left(\text{or } \frac{\pi}{3}\right)$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |

Special angles are any multiple of $30°$ $\left(\frac{\pi}{6}\right)$ or $45°\left(\frac{\pi}{2}\right)$. To find trig functions of any of these angles, draw them and find the **reference angle** (the angle created with the *x*-axis). Although most problems in calculus will use radians, you might think easier using degrees. This will create one of the triangles above and trig functions can be found, remembering to include the sign based on the quadrant of the angle. Finally, if an angle is outside the range of $0°$ to $360°(0 \text{ to } 2\pi)$, you can always add or subtract $360°(2\pi)$ to find trig functions of that angle. These angles are called **co-terminal angles**. It should be pointed out that $390° \neq 30°$ but $\sin 390° = \sin 30°$.

• Find the exact value of the following

a. $4\sin 120° - 8\cos 570°$

Subtract $360°$ from $570°$
$4\sin 120° - 8\cos 210°$
$120°$ is in quadrant II with reference angle $60°$.
$210°$ is in quadrant III with reference angle $30°$.
$4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{-\sqrt{3}}{2}\right) = 6\sqrt{3}$

b. $\left(2\cos\pi - 5\tan\frac{7\pi}{4}\right)^2$

$\left(2\cos 180° - 5\tan 315°\right)^2$
$180°$ is a quadrant angle
$315°$ is in quadrant III with reference angle $45°$
$\left[2(-1) - 5(-1)\right]^2 = 9$

**Q. Special Angles – Assignment**

• Evaluate each of the following without looking at a chart.

1. $\sin^2 120° + \cos^2 120°$

2. $2\tan^2 300° + 3\sin^2 150° - \cos^2 180°$

3. $\cot^2 135° - \sin 210° + 5\cos^2 225$

4. $\cot(-30°) + \tan(600°) - \csc(-450°)$

5. $\left(\cos\frac{2\pi}{3} - \tan\frac{3\pi}{4}\right)^2$

6. $\left(\sin\frac{11\pi}{6} - \tan\frac{5\pi}{6}\right)\left(\sin\frac{11\pi}{6} + \tan\frac{5\pi}{6}\right)$

• Determine whether each of the following statements are true or false.

7. $\sin\frac{\pi}{6} + \sin\frac{\pi}{3} = \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$

8. $\dfrac{\cos\frac{5\pi}{3} + 1}{\tan^2\frac{5\pi}{3}} = \dfrac{\cos\frac{5\pi}{3}}{\sec\frac{5\pi}{3} - 1}$

9. $2\left(\frac{3\pi}{2} + \sin\frac{3\pi}{2}\right)\left(1 + \cos\frac{3\pi}{2}\right) > 0$

10. $\dfrac{\cos^3\frac{4\pi}{3} + \sin\frac{4\pi}{3}}{\cos^2\frac{4\pi}{3}} > 0$

**R. Trigonometric Identities**

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier. The following chart gives the major trig identities that you should know. To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

Fundamental Trig Identities

$$\csc x = \frac{1}{\sin x},\quad \sec x = \frac{1}{\cos x},\quad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x},\quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1,\quad 1+\tan^2 x = \sec^2 x,\quad 1+\cot^2 x = \csc^2 x$$

• Verify the following identities.

1. $\left(\tan^2 x+1\right)\left(\cos^2 -1\right) = -\tan^2 x$

$$\left(\sec^2 x\right)\left(-\sin^2 x\right)$$
$$\left(\frac{1}{\cos^2 x}\right)\left(-\sin^2 x\right)$$
$$-\tan^2 x$$

2. $\sec x - \cos x = \sin x \tan x$

$$\frac{1}{\cos x} - \cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$$
$$\sin x\left(\frac{\sin x}{\cos x}\right) = \sin x \tan x$$

3. $\dfrac{\cot^2 x}{1+\csc x} = \dfrac{1-\sin x}{\sin x}$

$$\left(\frac{\frac{\cos^2}{\sin^2 x}}{1+\frac{1}{\sin x}}\right)\frac{\sin^2 x}{\sin^2 x} = \frac{\cos^2 x}{\sin^2 x + \sin x}$$
$$\frac{1-\sin^2 x}{\sin x(1+\sin x)} = \frac{(1+\sin x)(1-\sin x)}{\sin x(1+\sin x)}$$
$$\frac{1-\sin x}{\sin x}$$

4. $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$

$$\left(\frac{1+\sin x}{\cos x}\right)\left(\frac{1+\sin x}{1+\sin x}\right) + \left(\frac{\cos x}{1+\sin x}\right)\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1+2\sin x+\sin^2 + \cos^2 x}{\cos x(1+\sin x)}$$
$$\frac{1+2\sin x+1}{\cos x(1+\sin x)} = \frac{2+2\sin x}{\cos x(1+\sin x)}$$
$$\frac{2(1+\sin x)}{\cos x(1+\sin x)} = 2\sec x$$

**R. Trig Identities – Assignment**

- Verify the following identities.

1. $(1+\sin x)(1-\sin x)=\cos^2 x$

2. $\sec^2 x+3=\tan^2 x+4$

3. $\dfrac{1-\sec x}{1-\cos x}=-\sec x$

4. $\dfrac{1}{1+\tan x}+\dfrac{1}{1+\cot x}=1$

5. $\dfrac{\cos x-\cos y}{\sin x+\sin y}+\dfrac{\sin x-\sin y}{\cos x+\cos y}=0$

6. $\dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}=1-\sin x\cos x$

**S. Solving Trig Equations and Inequalities**

Trig equations are equations using trig functions. Typically they have many (or infinite) number of solutions so usually they are solved within a specific domain. Without calculators, answers are either quadrant angles or special angles, and again, they must be expressed in radians.

For trig inequalities, set both numerator and denominator equal to zero and solve. Make a sign chart with all these values included and examine the sign of the expression in the intervals. Basic knowledge of the sine and cosine curve is invaluable from section R is invaluable.

• Solve for $x$ on $[0,2\pi)$

1. $x\cos x = 3\cos x$

Do not divide by $\cos x$ as you will lose solutions

$\cos x(x-3)=0$

$\cos x = 0 \qquad x-3=0$

$x=\frac{\pi}{2},\frac{3\pi}{2} \qquad x=3$

You must work in radians.

Saying $x=90°$ makes no sense.

2. $\tan x+\sin^2 x = 2-\cos^2 x$

$\tan x+\sin^2 x+\cos^2 x=2$

$\tan x+1=2$

$\tan x=1$

$x=\frac{\pi}{4},\frac{5\pi}{4}$

Two answers as tangent is positive in quadrants I and III.

3. $3\tan^2 x-1=0$

$3\tan^2 x=1$

$\tan^2 x=\frac{1}{3}$

$\tan x=\pm\sqrt{\frac{1}{3}}=\pm\frac{\sqrt{3}}{3}$

$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

4. $3\cos x=2\sin^2 x$

$3\cos x=2\left(1-\cos^2 x\right)$

$2\cos^2 x+3\cos x-2=0$

$(2\cos x-1)(\cos x+2)=0$

$2\cos x=1 \qquad \cos x=-2$

$\cos x=\frac{1}{2} \qquad$ No solution

$x=\frac{\pi}{3},\frac{5\pi}{3}$

7. Solve for $x$ on $[0,2\pi)$ : $\frac{2\cos x+1}{\sin^2 x}>0$

$2\cos x=-1 \Rightarrow \cos x=\frac{-1}{2} \Rightarrow x=\frac{2\pi}{3},\frac{4\pi}{3}$

$\sin^2 x=0 \Rightarrow x=0,\pi$

Answer: $\left[0,\frac{2\pi}{3}\right)\cup\left(\frac{4\pi}{3},2\pi\right)$

+++++++0 −−−−−−− ∞ −−−−−−− 0+++++++

| $0$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|

## S. Solving Trig Equations and Inequalities - Assignment

• Solve for $x$ on $[0,2\pi)$

1. $\sin^2 x = \sin x$

2. $3\tan^3 x = \tan x$

3. $\sin^2 x = 3\cos^2 x$

4. $\cos x + \sin x \tan x = 2$

5. $\sin x = \cos x$

6. $2\cos^2 x + \sin x - 1 = 0$

7. Solve for $x$ on $[0,2\pi)$: $\dfrac{x-\pi}{\cos^2 x} < 0$

## T. Graphical Solutions to Equations and Inequalities

You have a shiny new calculator. So when are we going to use it? So far, no mention has been made of it. Yet, the calculator is a tool that is required in the AP calculus exam. For about 25% of the exam, a calculator is permitted. So it is vital you know how to use it.



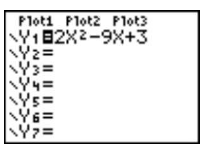
There are several settings on the calculator you should make. First, so you don't get into rounding difficulties, it is suggested that you set your calculator to three decimal places. That is a standard in AP calculus so it is best to get into the habit. To do so, press MODE and on the $2^{nd}$ line, take it off FLOAT and change it to 3. And second, set your calculator to radian mode from the MODE screen. There may be times you might want to work in degrees but it is best to work in radians.

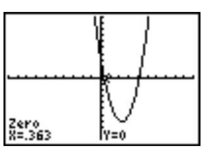
You must know how to graph functions. The best way to graph a function is to input the function using the Y= key. Set your XMIN and XMAX using the WINDOW key. Once you do that, you can see the graph's total behavior by pressing ZOOM 0. To evaluate a function at a specific value of $x$, the easiest way to do so is to press these keys: VARS → 1:Function 1 1:Y1 ( and input your $x$-value.
Other than basic calculations, and taking trig functions of angles, there are three calculator features you need to know: evaluating functions at values of $x$ and finding zeros of functions, which we know is finding where the function crosses the $x$-axis. The other is finding the point of intersection of two graphs. Both of these features are found on the TI-84+ calculator in the CALC menu $2^{ND}$ TRACE. They are 1:value, 2: zero, and 5: intersect.

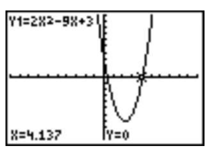
Solving equations using the calculator is accomplished by setting the equation equal to zero, graphing the function, and using the ZERO feature. To use it, press $2^{ND}$ TRACE ZERO . You will be prompted to type in a number to the left of the approximate zero, to the right of the approximate zero, and a guess (for which you can press ENTER). You will then see the zero (the solution) on the screen.

- Solve these equations graphically.

1. $2x^2 - 9x + 3 = 0$

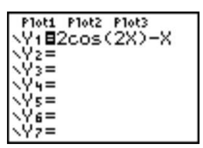


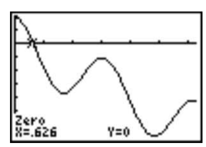



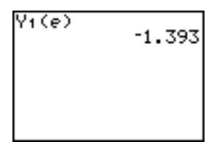




This equation can be solved with the quadratic formula.
$$x = \frac{9 \pm \sqrt{81-24}}{4} = \frac{9 \pm \sqrt{57}}{4}$$

2. $2\cos 2x - x = 0$ on $[0, 2\pi)$ and find $2\cos(2e) - e$.



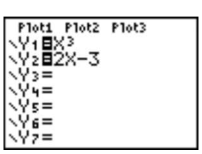



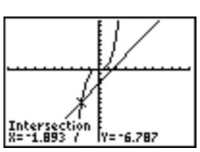




If this were the inequality $2\cos 2x - x > 0$, the answer would be $[0, 0.626)$.

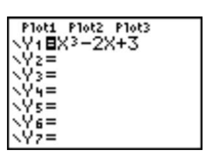
3. Find the $x$-coordinate of the intersection of $y = x^3$ and $y = 2x - 3$

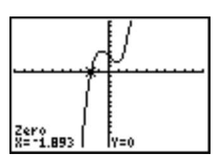
You can use the intersection feature.




Or set them equal to each other: $x^3 = 2x - 3$ or $x^3 - 2x + 3 = 0$



**T. Graphical Solutions to Equations and Inequalities – Assignment**

• Solve these equations or inequalities graphically.

1. $3x^3 - x - 5 = 0$

2. $x^3 - 5x^2 + 4x - 1 = 0$

3. $2x^2 - 1 = 2^x$

4. $2\ln(x+1) = 5\cos x \quad \text{on } [0, 2\pi)$

5. $x^4 - 9x^2 - 3x - 15 < 0$

6. $\dfrac{x^2 - 4x - 4}{x^2 + 1} > 0 \text{ on } [0, 8]$

7. $x\sin x^2 > 0 \text{ on } [0, 3]$

8. $\cos^{-1} x > x^2 \quad \text{on } [-1, 1]$